Dynamic geometry-based constructions for angle trisection and select non-constructible regular polygons: an alternative to classical methods

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Abstract

This paper presents an original method of angle trisection using dynamic geometry as an extension of the classical straightedge-and-compass framework. By incorporating variations in geometric parameters such as segment lengths and angle measures, the method circumvents the classical impossibility constraints without violating the foundational tools. The trisection process relies on geometric constructions derived from trigonometric identities and is validated through manual replication and iterative adjustment. While describing historical attempts and methods that were found inapplicable under the strict constraints of Greek mathematicians, this paper develops a dynamic approach that enables the trisection of angles that are proven non-constructible in static geometry. It also facilitates the construction of regular polygons (e.g., non-constructible 9- and 18-gons) whose defining angles are otherwise inaccessible. Importantly, this method does not resolve the classical Greek problem of angle trisection in its strictest form; rather, it offers an alternative constructive pathway while the original problem remains formally unsolved.

1. **Introduction**

It occupied the minds of ancient Greek mathematicians to double the unit cube, trisect an angle, and square a circle geometrically with the use of a straightedge and compass in the interest of purity of geometry, but these constructions could not be accomplished and still remain unconstructible. As Heath discusses, the origins of these three problems from antiquity are not well-documented [5]. This is likely attributed to how early Greek mathematical discoveries were orally transmitted before they were formally recorded—a characteristic feature of the Greek intellectual tradition at the time [10] [3]. This also contributed to the deceptive simplicity that made them enduring challenges. Although dividing an angle into two equal parts (bisection) is straightforward, the Greeks found trisecting a given angle nearly impossible within their geometric constraints. Certain angles, such as $2\pi, \pi, \pi/2, \pi/4, 3\pi/20$ radians are trisectable, but this property is not universal.

We say that a geometric figure is constructible if it can be drawn using only an unmarked straightedge and compass. An unmarked straightedge, as shown in Figure 1, is a flat, metallic or wooden straight piece that does not have any marks written upon it and thus can not be used for measuring length. However, it can be used to draw straight lines, join points, extend the straight lines, and compare their lengths. A compass is a geometric tool, as shown in Figure 2, that has one pointed end to be fixed to the paper and the other end has the facility to hold a pencil or a marker. Its two ends can be opened according to the desired width. In modern geometric tools, the adjustment of the compass once made can be locked to avoid any change in it while constructing a geometric figure, but in Euclidean geometry, the compass used to collapse once it was lifted from the paper. A compass is used to draw circles, compare radii, bisect angles and line segments, and mark equal lengths. It can also be used to compare distances by adjusting its opening. However, it has no markings to show the actual measurement of angles.

Figure 1 An ornate 18th century iron straightedge Figure 2 A Compass Straightedge Wikimedia under CCO 1.0 licence

According to Gauss theorem angle $2\pi/N$, where N is a positive integer, is constructible if N is given by Equation (1.1). In this regard, I refer to Fermat primes denoted by $p_1, p_2, p_3, \ldots,$ which ascertain feasibility of construction of an angle $2\pi/N$ that in fact is an external angle of N equal sided polygon or N sided regular polygon. Fermat primes are specific integers of the form $2^{2^k}+1$, where k is a non-negative integer including zero. While the first few Fermat numbers (e.g., 3, 5, 17) are prime, higher Fermat numbers, such as $F_5 = 4{,}294{,}967{,}297$, are composite. Mathematically, an angle 2π/N is constructible when

$$N = 2^m (p_1)(p_2)(p_3) \ldots (p_M), \tag{1.1}$$

$p_1, p_2, p_3, \ldots, p_M$ are distinct Fermat's primes, $m = 0, 1, 2, \ldots$ [3].

Coming to trisectability, if $N = (2^m)(p_2)(p_3) \ldots (p_M)$, angle $2\pi/N$ is constructible, and its trisected angle $2\pi/(3N)$ is also constructible, since number 3 is also a Fermat prime. But when N instead of being equal to $(2^m)(p_2)(p_3) \ldots (p_M)$, is equal to $3(2^m)(p_2)(p_3) \ldots (p_M)$, therefore, it would not be trisectable, because in the trisection, N would equal $(3)(3)(2^m)(p_2)(p_3) \ldots (p_M)$, which contains a repeated factor of 3 and thus violates the condition of Equation (1.1) [3]. Concluding it, for an angle 2π/N to be intrinsically trisect-able, N must not be divisible by 3, and the angle must also be constructible. Notwithstanding the above condition, a derivation of the Galois theory provides that an angle T is constructible if and only if the complex number e^{iT} lies in a field extension of rationals of degree a power of 2.

1.1 Historical Context

The question arises as to why Greek mathematicians were particular about the constructibility of geometric figures using a straight edge and compass. Its answer lies in the purity of geometrical construction with tools as simple as a straightedge and a compass. 'Purity' thus denotes adherence to geometric principles in methods that trisect a given angle.

Coming to trisectability, some angles using Fermat's primes as stated above are trisectable geometrically, but all angles are not trisect-able. The impossibility stems from the fact that, unlike quadratic equations, which can be solved using geometric construction, a general cubic equation cannot be solved with just a straightedge and compass. This limitation directly translates into the classical Delian problem of doubling the cube—a task proven impossible by Pierre Wantzel in 1837 [10].

Nevertheless, early mathematicians such as Hippocrates, Archimedes and others made notable attempts to solve this problem, often resorting to methods that violated the classical geometric rules. Hippocrates used a geometric drawing as given in Figure 3 which is explained herein under.

1.1.1 Hippocrates Geometric Construction

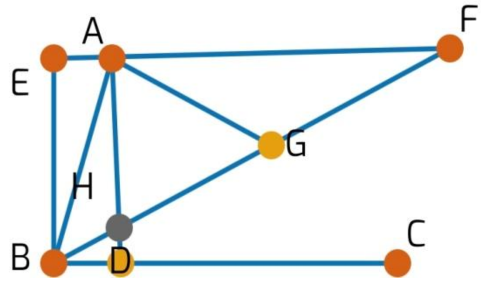


Figure 3 Angle Trisection by Hippocrates Construction

Construction: To trisect ∠ABC, construct $\overline{AD} \perp \overline{BC}$, where D is on $\overline{BC}$. Complete the rectangle ADBE. Construct $\overrightarrow{EA}$ with F beyond A on the ray. Draw $\overline{BF}$ passing through H, which is the intersection of $\overline{AD}$ and $\overline{BF}$ such that $\overline{\mathrm{HF}} = 2\ \overline{\mathrm{AB}}$

By construction, $\mathrm{m}\angle\mathrm{FBC} = \frac{1}{3}\mathrm{m}\angle\mathrm{ABC}$ [8].

Proof:

1. Mark G as the midpoint of $\overline{HF}$.
2. Construct $\overline{AG}$
3. Δ FAH has a right angle at A by construction, therefore $\overline{\mathrm{AG}} = \overline{\mathrm{HG}} = \overline{\mathrm{GF}}$.
4. By construction, $\overline{AB} = \frac{1}{2}\overline{HF}$. By substitution, $\overline{AB} = \overline{AG} = \overline{HG} = \overline{GF}$.
5. Δ ABG is an isosceles triangle, thus $\mathrm{m}\angle\mathrm{ABG} = \mathrm{m}\angle\mathrm{AGB}$.
6. Δ GAF is an isosceles triangle, thus $\mathrm{m}\angle\mathrm{GAF} = \mathrm{m}\angle\mathrm{GFA}$.
7. In Δ GAF, use the external angle relationship, $\mathrm{m}\angle\mathrm{BGA} = \mathrm{m}\angle\mathrm{GAF} + \mathrm{m}\angle\mathrm{AFG}$. Since the triangle is isosceles, $\mathrm{m}\angle\mathrm{BGA} = 2(\mathrm{m}\angle\mathrm{AFG})$.
8. By construction, $\overline{\mathrm{EF}} \parallel \overline{\mathrm{BC}}$. Additionally, $\mathrm{m}\angle\mathrm{CBF} = \mathrm{m}\angle\mathrm{AFG}$ as they are alternate angles of parallel lines.
9. Consider then $\angle ABC$. By the angle addition postulate, $m\angle ABC = m\angle CBF + m\angle GBA$. By substitution, $m\angle ABC = m\angle CBF + m\angle BGA$. Again, by substitution, $m\angle ABC = m\angle CBF + 2(m\angle AFG)$. By substitution once more, $\mathrm{m}\angle\mathrm{ABC} = \mathrm{m}\angle\mathrm{CBF} + 2(\mathrm{m}\angle\mathrm{CBF})$.
10. Therefore, $m\angle ABC = 3(m\angle CBF)$, and the angle is trisected.

While this construction is mechanically correct, it uses a marked ruler, violating classical constraints. As Plato remarked: "In proceeding in [a mechanical] way, did not one lose irredeemably the best of geometry [8]?"

Another mechanical solution found in Arabic work in the 'Book of Lemmas' is attributed to Archimedes. In fact, this work is not a translated copy of the work of Archimedes, but most historians believe much of the work given in the said book belongs to Archimedes [8].

1.1.2 Archimedes' Construction

To trisect $\angle DFA$, construct a circle with centre F and points A and D on its circumference. Draw a line from D to intersect the circle at C and the extended line AF at B, ensuring $\overline{\mathrm{CB}} = \overline{\mathrm{CF}}$. Construct $\overline{\mathrm{EF}} \parallel \overline{\mathrm{DB}}$, ensuring $\overline{\mathrm{CF}} = \overline{\mathrm{EF}} = \mathrm{r}$, the radius of the circle.

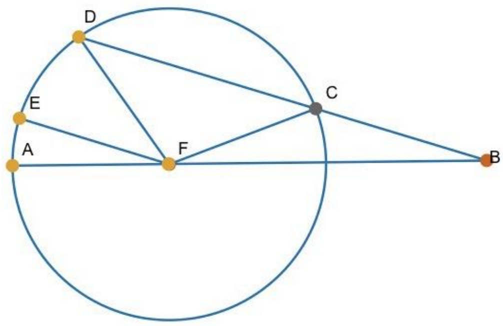


Figure 4 Angle Trisection by Archimedes' Construction

Proof:

i. In isosceles $\triangle\, CFB$, $m\angle CFB = m\angle CBF$, therefore, external $m\angle DCF = 2(m\angle CBF)$.

ii. In a triangle $\triangle\, FCD$, $m\angle DCF = m\angle FDC$, since $\overline{\mathrm{FC}}$, $\overline{\mathrm{FD}}$ are the radii of the circle.

iii. The $m\angle CDF = m\angle DEF$, being alternate angles, therefore, $m\angle DEF = 2(m\angle CBF)$.

iv. The $m\angle CFB = m\angle AFE$, being corresponding angles, therefore,$m\angle AFE = \frac{1}{2}(m\angle EFD$ or $m\angle AED = \left(\frac{1}{3}\right) m\angle AFD$ [8].

As with Hippocrates' method, this construction relies on mechanical aids, making it unacceptable under classical geometric rules. Another method, though not yielding accurate trisection, is based on summation of an infinite series whereby the sum of some of the terms of the series was used. Another method was given by Ludwig Biebetbach but it used trisectrix and was not acceptable [1]. A trisectrix is a plane curve such that, for a given angle, a point moving along the curve allows the construction of one-third of that angle.

1.1.2 Trisectrix of Maclaurin

The Trisectrix of Maclaurin is the locus of the intersection point of two lines rotating about their centres $(0,0)$ and $(a,0)$ at uniform angular speeds, with the line rotating about $(a,0)$ moving three times as fast as the line rotating about (0, 0). Initially, both lines coincide along the segment connecting(0,0) and $(a, 0)$. The curve traced forms a loop that extends to infinity both upward and downward, with $x = -a/2$ as its asymptote [11]. It intersects the x-axis at $(0,0)$ and $(3a/2, 0)$, and intersects the vertical line $x = a$ at $y = \pm a/3^{1/2}$ [11]. The $m\angle OQP = \frac{2}{3}(m\angle\varphi)$, and the distance $r = \overline{OP}$ is determined by the law of sines:

$$r/sin(\varphi) = a/sin(2\varphi/3).$$

If the given, $m\angle QPX = m\angle\varphi$, then trisected angle is QOX $(\varphi/3)$ [11].

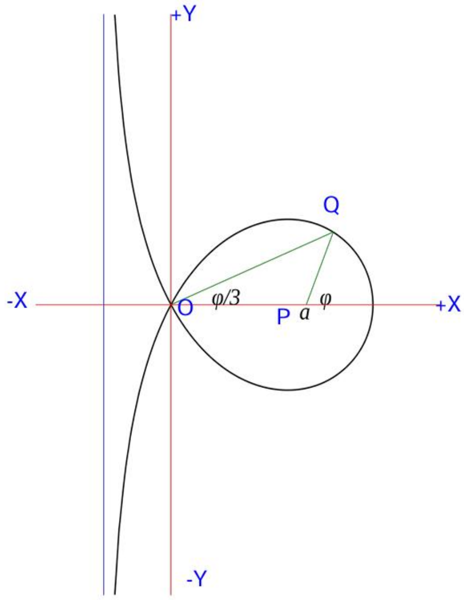


Figure 5 Trisectrix of Maclaurin, Image: MaclaurinTrisectrix.SVG by RDBury, created using gnuplot and Inkscape

2 Modern Approaches and Use of Physical and Intuitive Methods

2.1 Angle Trisection by Paper Folding Art Origami

Cubic equations including $cos(x) = 4\,cos^3(x) - 3\,cos\,(x)$ where angle x is to be trisected, are solvable using Origami [4], [6] by folding a sheet of paper along a straight line such that a set of particular incidences is obtained between points and lines. These incidences are determinable by the coefficients of the equation.

Consider an angle of $\pi/3$ radian, which is to be trisected, relates to solving a polynomial equation $P(cos\,x) = 0,$ or $4\,cos^3(x) - 3\,cos\,(x) - 1/8 = 0.$ By the rational root theorem, this equation can have rational roots, $\pm1, \pm1/2, \pm1/4, \pm1/8.$ But none of these happen to be a root of the equation. Therefore, the equation is not reducible over rational roots, and the minimum polynomial has a degree of three. Thus, the angle π/3 is not trisect-able.

2.2 Trisection of an Arbitrary Angle by Dynamic Geometry

2.2.1 Dynamic Geometry

A geometric construction ordinarily is drawn according to the given conditions. I call such a drawing as belonging to Static Geometry since no adjustment in length or angle is permissible and these are drawn, according to the given geometric statement. At times, given conditions can not be fulfilled by geometric drawing and need variation in length, angle or other geometric elements bringing in dynamic approach. I. M. Yaglom in his book Geometric Transformations stresses that interactive and real-time manipulation is vital for solving non-conventional geometric problems [9].

Such a dynamic approach brings in use of dynamic geometry. Dynamic geometry is a branch of geometry that focuses on constructing geometric figures based on given conditions and exploring how variations in elements, such as angles, sides, or points, affect the figure's ability to meet those conditions. Unlike traditional static constructions, dynamic geometry encourages interactive exploration, where the geometric objects can be manipulated in real-time. This interactivity allows users to investigate geometric properties and solve problems that may not be easily approached through conventional methods, such as using a straightedge and compass.

The use of software tools like Geometer'sSketchpad, GeoGebra, or Cabri Geometry significantly enhances the ability to explore, visualise, and manipulate geometric figures. Dynamic geometry provides an interactive environment that facilitates the understanding of geometric relationships and the discovery of solutions to complicated problems. By engaging with dynamic geometry, users are able to investigate geometric configurations more deeply and intuitively, making it a valuable tool for both learning and solving advanced geometric problem.

2.2.2 Dynamic Geometry: Alternative Constructions Within Classical Boundaries

Dynamic geometry offers an alternative paradigm that operates within a different conceptual and operational framework from classical compass-and-straightedge constructions. It does not contradict classical impossibility theorems—such as the trisection of an arbitrary angle or the duplication of the cube—but instead circumvents their constraints by leveraging continuous motion, loci, and real-time feedback mechanisms. These tools allow constructions that are not permitted in the static framework of classical geometry, thus opening new pedagogical and exploratory avenues without invalidating the foundational results of traditional Euclidean theory.

2.2.3 Construction of Trisection of an Arbitrary Angle

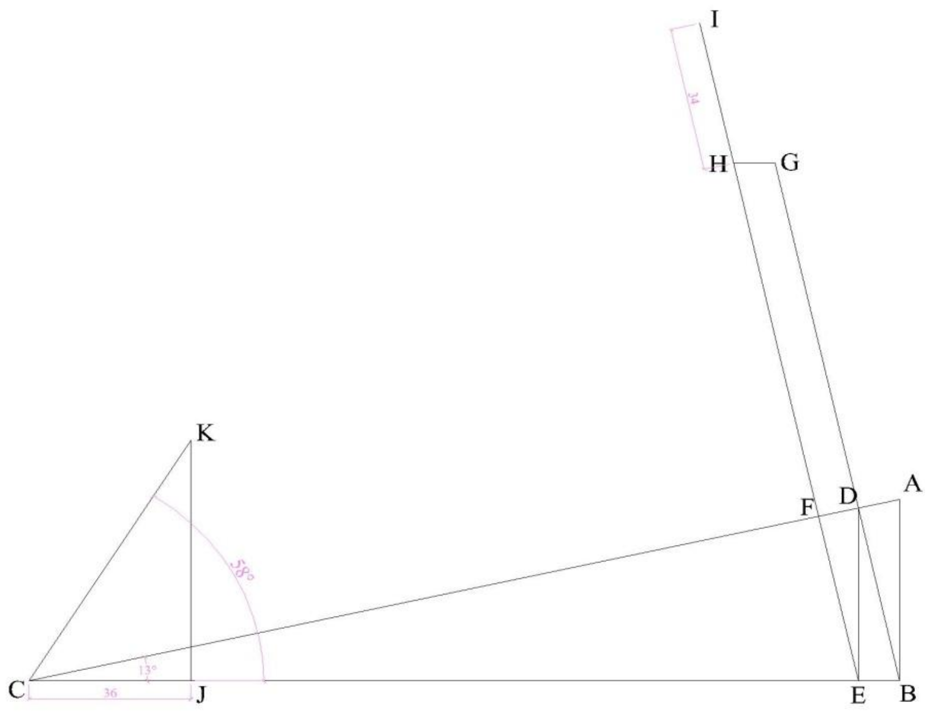


Figure 7 Uncorrected Angle Trisection

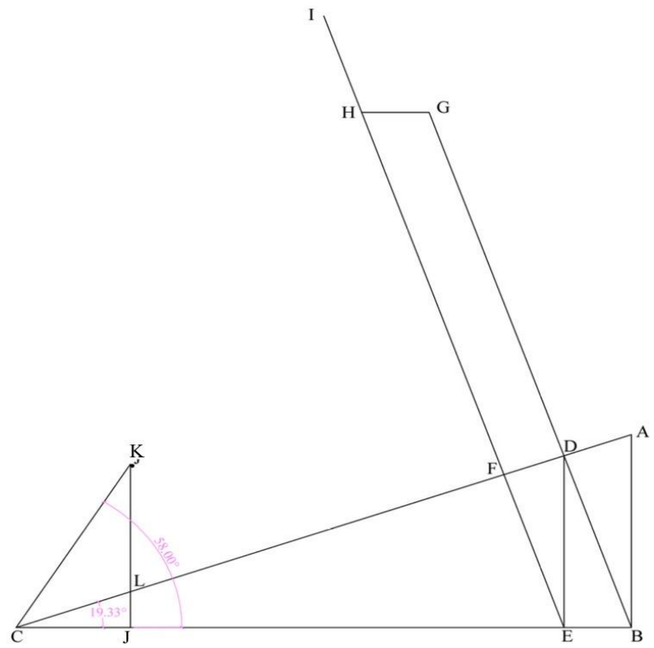


Figure 8 Corrected Angle Trisection

The given $\angle KCB$ is shown in Figure 7, a line segment CK equal to unity is marked, and from point $K,$ construct $\overline{KJ} \perp \overline{CB}$, meeting it at J.

i. A ΔABC is constructed with an arbitrary base angle at point C and $\overline{AB}$ is of unit length and is $\perp$ line CB.

ii. From point B, construct $\overline{BD} \perp \overline{AC}$, from point D, construct $\overline{DE} \perp \overline{BC}$ and from point E, construct $\overline{EF} \perp \overline{AC}$.

iii. Perpendiculars BD is extended to point $G,$ so that $\overline{BG} = 3\overline{BD},$ similarly the line segment $\overline{EF}$ is extended to H, so that $\overline{EH} = 4\overline{EF}$.

iv. From point G, a line GH is drawn so that $\text{m}\angle GHI = \text{m}\angle IEB$.

v. The compass is opened equal to length segment CJ and compared with length of line segment HI. If both lengths are equal, no further change in construction is needed.

vi. Figure 7 illustrates the uncorrected construction where $\text{m}\angle C(\text{m}\angle ACB)$ takes an arbitrary value, resulting in unequal lengths CJ and HI. Refer to Figure 7, $CJ = 36$ and $HI = 34$ units. This inequality reflects the error in trisection when the cosine triple-angle identity is applied without adjustment. Kindly note inequality in trisection in Figure 7 as $\text{m}\angle KCB = 58$ degrees and$\text{m}\angle ACB = 13$ degrees. Iterative corrections are performed by varying $\angle C$ along extended length BJ, as detailed in steps iii to v, until $\overline{CJ}$ equals $\overline{HI}$. Kindly peruse corrected construction in Figure 8.

vii. Figure 8 displays the corrected construction where $\text{m}\angle C(\text{m}\angle ACB)$ has been adjusted to achieve exact triection. This is confirmed when $\overline{CJ}$ equals $\overline{HI}$, both measuring 36 units, and the angles are verified using AutoCAD: $\text{m}\angle ACB = 19.33$ degrees and$\text{m}\angle KCB = 58$ degrees. Please note that, AutoCAD software application was used only for the purpose of clarifying the equality in lengths and angle trisection; otherwise the constructions have been achieved using a straight edge and compass, employing dynamic geometry.

Proof of Trisection of Angle

i. Referring to Figure 8, in $\Delta\ KCJ, \overline{CK} = 1$ unit, therefore, $\overline{CJ} = cos(\angle KCJ)$.

ii. In ΔABD, m$\angle ABD =$ m$\angle C$, therefore $\frac{BD}{AB} = \cos(C)$ or $\overline{BD} = \cos$ (C), since $\overline{AB} = 1$.

iii. In triangle DBE, $DE/DB = cos(C)$ or $\overline{DE} = cos^2(C)$, since $\overline{BD} = \cos$ (C).

iv. In triangle DEF, $EF/DE = \cos$) or $\overline{EF} = cos^3\ (C)$, since $\overline{DE} = cos^2(C)$.

v. Since $HI = HE - GB = 4cos^3(C) - 3cos(C)$, therefore, $HI = cos(3C)$.

vi. When $\overline{CJ} = \overline{HI}$, then $\cos\left(\frac{1}{3}\angle \text{KC}B\right) = \cos\angle ACB)$ or m$\angle KCB = 3($m$\angle$ACB).

vii. Hence base angle m$\angle ACB$or m$\angle C = \frac{1}{3}$(m$\angle$KCB).

3. Construction of Regular Polygons Based on Angle Trisection

3.1 External and Internal Angles of a Regular N Sided Polygon

A regular polygon is a geometric figure that has n equal sides, where n can be $3, 4, 5$ ... A straight line subtends an angle of πradians. If vertex angle A of the triangle ABC is enlarged, its base angles will decrease, eventually reaching zero when angle A attains π radians. This leads to the result that sum of three angles that its sides subtend equals to π radians. A natural corollary is that when its sides are equal, all three angles are equal with angle $\pi/3$ radians in an equilateral triangle (or a regular polygon of 3 sides).

When the base BC of the triangle ABC, is extended, each of its external angle becomes $(\pi - \pi/3) = 2\pi/3$ radians. Similarly for a polygon with four sides each external angle is $2\pi/4$ radians. By following the same procedure, the external angle of a regular polygon with n sides is $2\pi/n$.Consequently, its internal angle will be$\pi - 2\pi/n$.

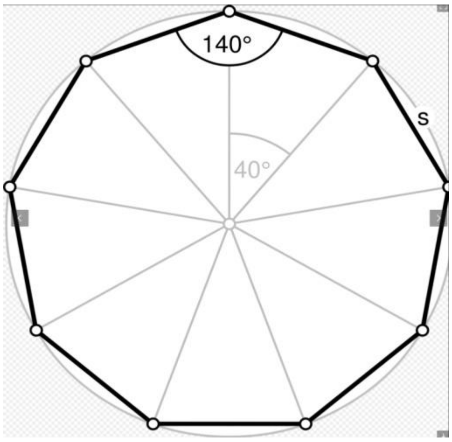
That leads to the conclusion, if an angle $2\pi/n$ is constructible, then a regular polygon with n sides is also constructible. However, not all regular polygons can be constructed using a straightedge and compass. Referring to Section 1, an angle $2\pi/n$ is constructible if and only if n is a power of 2 or a product of powers of 2 and one or more Fermat primes.

3.2 Construction of Non-Constructible Regular Polygons

If q is a positive integer and n is of type $3q$ satisfying Equation (1.1), then a regular polygon of sides $3q$ is constructible, but a regular polygon of sides 3^2q is not constructible as $n = (3)(3)q$ does not have distinct Fermat's primes. As explained above, dynamic geometry facilitates construction of a non constructible angle by its trisection, therefore, trisected external angle $2\pi/\{(3)(3)q\}$ although is non constructible using a straightedge and compass, is made constructible employing dynamic geometry, facilitates construction of non constructible regular polygons. Thus use of dynamic geometry helps geometric construction of non constructible regular polygons of sides $\text{n} = 2^{\text{k}}(3^{\text{r}})(\text{p}_1)(\text{p}_2)(\text{p}_3) \ldots (\text{p}_{\text{M}})$ where $\text{k}, \text{p}_1, \text{p}_2, \text{p}_3, \ldots, \text{p}_{\text{M}}$ have there unusual meaning and $\text{r} = 2, 3, 4, \ldots$ [2].

4. Construction of Non-Constructible Regular Polygons Using Dynamic Geometry

4.1 Construction of non-constructible 9-Sided and 18-Sided Regular Polygons



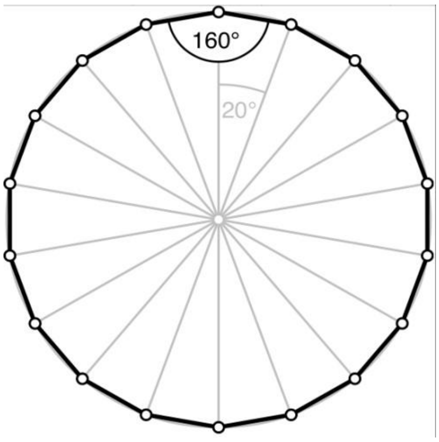


Figure 10 A 9-Sided Regular Polygon, Credit LászlóNémethCreative CommonsCC0 1.0 Universal

Figure 11 A 18-Sided Regular Polygon, Credit LászlóNémethCreative CommonsCC0 1.0 Universal

The external angle of 9-sided polygon as already discussed earlier is $2\pi/9$. Ordinarily, this angle is not constructible as it violates essential conditions of Equation (1.1) and it also corresponds to the equation $cos(2\pi/3) = 4cos^3(2\pi/9) - 3cos(2\pi/9)$ or $-1/2 = 4cos^3(2\pi/9) - 3cos(2\pi/9)$ which does not satisfy the constructibility conditions using a straightedge and compass. However, this angle is constructed with angle trisection using dynamic geometry. Therefore, a regular 9-sided polygon is also constructible. Refer to Figure 10. Angle $2\pi/18$, is obtainedon further bisecting angle $2\pi/9$, and this angle $2\pi/18$corresponds to the external angle of an 18-sided polygon. Thus, an 18-sided regular polygon is also constructible.

5. Results and Conclusions

An angle, while bisect-able, cannot be trisected using only an unmarked straightedge and compass, as proven by Pierre Wantzel. However, an angle can be trisected geometrically using alternative methods, such as a marked straightedge and compass or mechanical means. These approaches were not approved by Greek mathematicians. Hippocrates' construction is feasible with a marked ruler but was dismissed by Greek mathematicians for relying on mechanical principles rather than purely geometric properties. Similarly, Archimedes' construction faced the same criticism. Approximation methods, such as the series expansion of $1/3$, lack exactness and fall outside the scope of classical geometry. Ludwig Bieberbach's use of a triangular ruler also exceeded the allowable tools, while Origami, though effective, similarly deviates from traditional geometric constraints [4], [6].

To address these limitations, this paper employed dynamic geometry. Unlike static geometry, where figures corresponding to given conditions are fixed, dynamic geometry allows variation in sides or angles to achieve constructions otherwise impossible with classical methods. While dynamic software tools are typically used, this study replicates their functionality through manual manipulation. As explained, the construction of dropping perpendiculars successively upon the hypotenuse and base of a right-angled triangle ABC, with a right angle at B and $\overline{AB}$ of unit length, yields the identity $EF/DE = DE/BD = BD/1 = \cos$. This further facilitates the relations, $\overline{BD} = \cos(C)$, $\overline{DE} = cos^2(C)$, $\overline{EF} = cos^3(C)$. As cosine of given angle equals base of another right-angled triangle assuming its hypotenuse of unit length can be equalised with length $4cos^3(C) - 3cos(C)$ by varying angle C and keeping $\overline{AB} = 1$, therefore, $\angle C$ is one third of the given angle facilitating the angle trisection.

This construction adheres to the use of a compass and an unmarked straightedge but relies on dynamic geometry principles rather than strict static geometry. The method incorporates the trigonometric identity for angle tripling in reverse. While the impossibility of angle trisection within static geometry rules remains valid, the use of dynamic geometry enables the construction.

The construction of an angle by trisection which otherwise is non constructible facilitates construction of non constructible regular polygons. Dynamic geometry facilitates construction of regular polygons that have external angles $2\pi/\{(2^k)(3^r)\}$ where $k = 0, 1, 2, \ldots$ and $r = 2, 3, 4, \ldots$ Further, if an n sided regular polygon with external angle $2\pi/n$ is constructible, by angle trisection using dynamic geometry, a polygon with external angle $2\pi/\{\mathrm{n}(2^{\mathrm{k}})(3^{\mathrm{m}})\}$ is also constructible.

Finally, the impossibility of trisecting an angle is further underscored by the observation that while a straight line can be divided into n equal parts using a straightedge and compass, an arc can only be divided into 2^n equal parts under classical geometric constraints. Looking at the arc as a line segment by straightening its curve, the resultant segment can be divided in three parts, and the parts can be marked. The line segment can then be restored to its curved form, the angle can be trisected by joining the marks of division at the angle point. Such construction is possible and has been utilised by Hutcheson [7], but this method also falls out of the conditions imposed by Greek mathematicians.

6. Supplementary Electronic Material

[S1] An HTML file titled "Interactive Angle Trisection" can be found **here**. For the trisection, the given $\angle KCB$ must be less than 90°, and point C should be slid along line CB until $\overline{CJ} = \overline{HI}$. Under this condition, the interactive figure displays $m\angle ACB = (1/3)(\mathrm{m}\angle KCB)$ showing the numerical values of both $\angle ACB$ and angle $(1/3)(\angle KCB)$.

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